Unboundable Spacetimes with Metric Singularities and Matching Metrics and Geodesics:

A Black-White Hole and a Big Crunch-Bang

Vladimir S. MASHKEVICH[1]

Queens College
The City University of New York
65-30 Kissena Boulevard
Flushing, New York 11367-1519

Abstract

Singularity theorems of general relativity utilize the notion of causal geodesic incompleteness as a criterion of the presence of a spacetime singularity. The incompleteness of a causal curve implies the end and/or beginning of the existence of a particle, which are events. In the commonly accepted approach, singularities are not incorporated into spacetime. Thus spacetime turns out to be event-incomplete. With creation from nothing, singularities are sources of lawlessness. A straightforward way around these conceptual problems consists in including metric singularities in spacetime and then matching metrics and causal geodesics at the singularities. To this end, a spacetime manifold is assumed to be unboundable, so that singularities may only be interior. The matching of the geodesics is achieved through weakening conditions for their smoothness. This approach is applied to a black-white hole and a big crunch-bang.

[1]E-mail: Vladimir_Mashkevich@qc.edu

Introduction

Spacetime singularities are inherent in general relativity, or more specifically in gravitational collapse and cosmology. The analysis of spacetimes with singularities is one of the most principal and difficult problems in general relativity. Singularity theorems of general relativity utilize the notion of causal geodesic incompleteness as a criterion for the presence of a singularity. (A comprehensive presentation and discussion is given in [1].) The incompleteness of a causal, i.e., timelike or null curve implies physically the end and/or beginning of the existence of a particle, which are undeniably events. In the commonly accepted approach, singularities are not incorporated into a spacetime manifold. Thus spacetime turns out to be event-incomplete, i.e., does not include all events.

Furthermore, the beginning and end of the existence of a free particle means that there is creation from nothing and extinction into nothing. Those phenomena are in conflict with conservation laws and appear physically pathological. Maybe it is possible to put up with extinction into nothing, arguing that nature is so structured. At least extinction follows a clear-cut law: Arriving at a singularity results in extinction. With creation from nothing, the situation is much worse. In this role, (naked) singularities are sources of lawlessness. All sorts of nasty things—green slime, Japanese horror movie monsters, etc.—may emerge helter-skelter from a singularity [1]. To get rid of that nightmare, Penrose proposed the cosmic censorship hypothesis. But cosmic censorship may be legislated only by a fiat, it does not follow from known physical laws.

Relying on the aforesaid, we state that there exists the conceptual problem of singularities, which should be dealt with. We shall restrict our consideration to metric singularities and their related causal geodesic incompleteness. A straightforward approach consists in including the singularities in spacetime and making them to be interior, not exterior ones, and then matching metrics and causal geodesics at the singularities. To this end, a spacetime manifold is assumed to be unboundable, i.e., to be a manifold without boundary and not to be a manifold with boundary, the latter being removed. Such a manifold is realized as a closed submanifold of a Euclidean space. The matching of metrics is based on matching tangents to curves through a singularity. The matching of causal geodesics is achieved through weakening conditions for their smoothness. The main remaining condition is that of cornerlessness.

The next step is made by considering a product spacetime. This allows one to formulate the conditions of smoothness in terms of time dependence of the geodesics. In particular, the examination of acceleration is helpful in matching geodesics.

The approach outlined above is applied to a black-white hole and a big crunch-bang. Synchronous coordinates are utilized. Radial geodesics are investigated.

The main results for black-white holes are as follows. There are two types of metric singularities: singular three-dimensional interfaces between black and white regions and singular threefolds within white regions. On the former surfaces particles reflect from hypersurfaces of constant radius, on the latter particles traverse through those.

For a big crunch-bang there is a metric-singular three-dimensional interface between contracting and expanding regions. On this surface particles traverse through hypersurfaces of constant radius.

1 A conventional concept of a singular spacetime and event-incompleteness of the latter

Our ultimate goal is to surmount causal geodesic incompleteness. The first step is to include singular points in a spacetime manifold. In a conventional treatment, such an inclusion is believed to be impossible. The main argument is that physical laws are violated at singular points. But if a set of the singular points is a hypersurface, the violation may be overcome through matching results of the laws at points of the hypersurface. On the other hand, excluding singular points from a spacetime manifold implies excluding corresponding events, which means event-incompleteness of spacetime. But spacetime is by definition the set of all events. Thus the inclusion of singularities in the spacetime manifold seems to be justified.

2 An unboundable manifold and a closed submanifold as its realization

Singularities not only have to be included in the spacetime manifold M, but also should not be localized on a boundary of M; for otherwise causal curves cannot be extended. The problem of boundary may be posed more generally. A boundary of M or a possibility of attaching one to M give rise to curve incompleteness. To prevent this we arrive at the notion of an unboundable manifold, i.e., a manifold without boundary to which no boundary can be attached:

$$M = (\text{manifold without boundary}) \text{ and } M \neq (\text{manifold with boundary}) - (\text{boundary}) \quad (2.1)$$

An unboundable manifold may be realized as a closed subset of an Euclidean space by the Whitney theorem [2]: A smooth manifold M^n can be embedded as a submanifold, and closed subset, of R^{2n+1}.

So we posit spacetime to be an unboundable manifold.

3 Interior singularities and matching metrics and geodesics

Interior singularities
Now that singularities may only be interior, the problem of overcoming causal geodesic incompleteness amounts not to extending a manifold and geodesics but rather to matching metrics and geodesics on opposite sides of a metric singularity hypersurface.

Matching metrics
First consider metrics. Let $p \in M$ be a point on a metric singularity hypersurface and $\gamma(u)$ be a C^∞ curve through this point such that

$$\gamma(u_{\text{sing}}) = p, \quad \dot\gamma(u_{\text{sing}}) \neq 0 \quad (3.1)$$

and $\dot\gamma(u_{\text{sing}})$ is not tangent to the hypersurface. Matching conditions are: for all γ specified

$$\int_{u_{\text{sing}}-\delta}^{u_{\text{sing}}+\delta} |g_{\gamma(u)}(\dot\gamma(u), \dot\gamma(u))|^{1/2} du < \infty \quad (3.2)$$

and

$$\lim_{\delta\to 0}\frac{g_{\gamma(u_{\rm sing}+\delta)}(\dot\gamma(u_{\rm sing}+\delta),\dot\gamma(u_{\rm sing}+\delta))}{g_{\gamma(u_{\rm sing}-\delta)}(\dot\gamma(u_{\rm sing}-\delta),\dot\gamma(u_{\rm sing}-\delta))}=1 \tag{3.3}$$

Matching geodesics
Now turn to geodesics. Geodesic equations are of the form

$$\frac{dK^\mu}{du}+\Gamma^\mu_{\nu\rho}K^\nu K^\rho=0,\quad K^\mu=\frac{dx^\mu}{du} \tag{3.4}$$

We put

$$du=\frac{1}{m}ds \tag{3.5}$$

for a particle of a mass $m\neq 0$. Then

$$K^\mu K_\mu=m^2 \tag{3.6}$$

holds for both $m\neq 0$ and $m=0$.

Let $x^\mu_{\rm sing}$ be coordinates of p and $x^\mu(u)$ be coordinates of two causal geodesics $\gamma(u)$, $u<u_{\rm sing}$ or $u>u_{\rm sing}$, on opposite sides of the singular hypersurface. Matching conditions are:

$$\lim_{\delta\to 0}\gamma(u_{\rm sing}+\delta)=\lim_{\delta\to 0}\gamma(u_{\rm sing}-\delta)=p \tag{3.7}$$

and

$$\lim_{\delta\to 0}\frac{(d^kx^\mu/du^k)_{u_{\rm sing}+\delta}}{(d^kx^\mu/du^k)_{u_{\rm sing}-\delta}}=1,\ k=1,2,\ldots,k_{\rm maximal} \tag{3.8}$$

$$\lim_{\delta\to 0}\left(\frac{d^kx^\mu}{du^k}\right)_{u_{\rm sing}\pm\delta}=+\infty\quad\text{or}\quad-\infty,\quad k=k_{\rm maximal} \tag{3.9}$$

being admissible. Such a curve may be termed a $\bar C^k$ curve (extended C^k curve).

4 A product spacetime

A product spacetime manifold is a typical and the most important instantiation of the notion of an unboundable spacetime manifold. The manifold $M=M^4$ is a direct product of two manifolds:

$$M=T\times S,\quad M\ni p=(t,s),\quad t\in T,\quad -\infty<t<\infty,\quad s\in S \tag{4.1}$$

The one-dimensional unboundable manifold T is time, the three-dimensional manifold S, which should be unboundable, is a space. By (4.1) the tangent space at a point $p\in M$ is

$$M_p=T_p\oplus S_p \tag{4.2}$$

Assuming that

$$T_p\perp S_p \tag{4.3}$$

it follows for the metric tensor that

$$g=g_T+g_S \tag{4.4}$$

Furthermore,

$$g = dt \otimes dt - h_t \tag{4.5}$$

where h_t is a Riemannian metric tensor on S depending on t. A relevant coordinate representation is

$$ds^2 = dt^2 - dh_t^2, \qquad dh_t^2 = h_{tij}^2 dx^i dx^j \tag{4.6}$$

$(t = x^0, x^1, x^2, x^3)$ are synchronous coordinates.

Now the problem of metric singularities amounts to that for the metric h. The problem is thereby greatly simplified since h is a Riemannian metric, for which there is a fully satisfactory notion of the location of singularities [3].

In the case of the metric (4.6), equation (3.6) reduces to

$$(K^0)^2 - h_{ij}K^iK^j = m^2 \tag{4.7}$$

or

$$\omega^2 - K_iK^j = m^2, \qquad \omega = E = K^0 \tag{4.8}$$

5 Matching geodesics: Refinement

Let us return to matching causal geodesics and employ the universal time t as a parameter. So a geodesic is given by functions $x^i(t)$. Let the geodesic pass through a singular point p. Choose coordinates so that $p = (0,0,0,0)$ and $x^2(t) = x^3(t) = 0$ along the geodesic for $t \approx 0$. Then the geodesic is described by a function $x(t) \equiv x^1(t)$. The geodesic is at least $\bar{C}^1$, so that we have for $t \approx 0$

$$|x(t)| \approx |x(-t)| \tag{5.1}$$

which implies

$$|\dot{x}(t)| \approx |\dot{x}(-t)|, \qquad |\ddot{x}(t)| \approx |\ddot{x}(-t)| \tag{5.2}$$

Furthermore, in view of singularity

$$\lim_{t\to 0} |\ddot{x}(t)| = \infty \tag{5.3}$$

There are two possibilities:

$$(1)\ \dot{x}(0) \neq 0,\ \lim_{t\to 0}|\dot{x}(t)| = \infty,\ \dot{x}(t) \approx \dot{x}(-t),\ x(t) \approx -x(-t),\ \ddot{x}(t) \approx -\ddot{x}(t),\ \mathrm{sgn}\ddot{x}(t) = -\mathrm{sgn}x(t) \tag{5.4}$$

this is the case of attraction, the particle transverses the point $x = 0$;

$$(2) \quad \dot{x}(0) = 0, \qquad \dot{x}(t) \approx -\dot{x}(-t), \qquad x(t) \approx x(-t), \qquad \ddot{x}(t) \approx \ddot{x}(-t), \qquad \mathrm{sgn}\ddot{x}(t) = \mathrm{sgn}x(t) \tag{5.5}$$

this is the case of repulsion, the particle reflects from the point $x = 0$.

6 A black-white hole

Spacetime manifold

Spacetime manifold of a spherically symmetric black-white hole is a product manifold (4.1), the space S being three-dimensional Euclidean space R^3:

$$M = T \times R^3 \tag{6.1}$$

Synchronous coordinates

Synchronous coordinates are t and spatial coordinates for R^3. In view of spherical symmetry, the spherical coordinates (R, θ, ϕ) are appropriate with $R = 0$ in the center of the star. Let the surface of the star correspond to $R = a$ ($a = \text{const}$ in synchronous coordinates). We are interested in the region outside the star, $R > a$.

Metric

The metric dh^2 is given as follows [4]. For $R > a$

$$dh^2 = \frac{[\partial_R r(t,R)]^2}{1+f(R)} dR^2 + r^2(t,R)(d\theta^2 + \sin^2\theta d\phi^2) \tag{6.2}$$

$$r = -\frac{r_g}{f(R)}\frac{1+\cos\eta}{2}, \quad t_0(R) - t = \frac{r_g}{[-f(R)]^{3/2}}\frac{\pi - \eta - \sin\eta}{2} \tag{6.3}$$

where $t_0(R)$ is an arbitrary function, $f(R)$ is an arbitrary function meeting the condition

$$-1 < f(R) < 0 \tag{6.4}$$

and r_g is the Schwarzschild radius. We put

$$t_0(R) = \frac{r_g}{[-f(R)]^{3/2}}\frac{\pi}{2} \tag{6.5}$$

so that

$$r = -\frac{r_g}{f(R)}\frac{1+\cos\eta}{2}, \quad t = \frac{r_g}{[-f(R)]^{3/2}}\frac{\eta + \sin\eta}{2} \tag{6.6}$$

Now we choose

$$f(R) = -\frac{r_g}{R} \tag{6.7}$$

In view of

$$R > a > r_g \tag{6.8}$$

the condition (6.4) holds. We obtain

$$r = R\frac{1+\cos\eta}{2}, \quad t = \frac{R^{3/2}}{r_g^{1/2}}\frac{\eta+\sin\eta}{2} \tag{6.9}$$

$$dh^2 = \frac{(\partial_R r)^2}{1 - r_g/R} dR^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{6.10}$$

For what follows, it is convenient to introduce quantities

$$\chi = \frac{1+\cos\eta}{2}, \qquad \xi = \frac{\eta+\sin\eta}{2} \tag{6.11}$$

so that

$$\chi = \chi(\xi), \qquad \xi = \frac{r_g^{1/2} t}{R^{3/2}}, \qquad r(t,R) = R\chi\left(\frac{r_g^{1/2} t}{R^{3/2}}\right) \tag{6.12}$$

the metric being given by (6.10).

Metric discontinuity on the surface of the star
Note that in synchronous coordinates, discontinuity of the matter density on the surface of the star results in a metric discontinuity as well, which is easily seen from [4,5].

Metric singularities
It follows from (6.11), (6.12) that

$$\partial_R r = \chi - \frac{3}{2}\xi\frac{d\chi}{d\xi} \tag{6.13}$$

$$\frac{d\chi}{d\xi} = -\frac{\sin\eta}{1+\cos\eta}, \qquad \left(\frac{d\chi}{d\xi}\right)^2 = \frac{1}{\chi} - 1 \tag{6.14}$$

$$\partial_R r = \frac{1+\cos\eta}{2} + \frac{3}{2}\frac{\sin\eta}{1+\cos\eta}\frac{\eta+\sin\eta}{2} \tag{6.15}$$

There are two types of metric singularities:

$$r = 0 \tag{6.16}$$

which implies

$$\chi = 0, \qquad \left(\frac{d\chi}{d\xi}\right)^2 = \infty, \qquad (\partial_R r)^2 = \infty \tag{6.17}$$

and

$$\partial_R r = 0 \tag{6.18}$$

which implies

$$\frac{3}{2}\eta\sin\eta + \frac{3}{2}\sin^2\eta + (1+\cos\eta)^2 = 0 \tag{6.19}$$

Let

$$t \geq 0 \tag{6.20}$$

(for example, $t = 0$ corresponds to the beginning of the collapse). Then

$$\xi \geq 0, \qquad \eta \geq 0 \tag{6.21}$$

For the singularities (6.16) we obtain

$$\eta = (2n+1)\pi,\ n = 0,1,2,\ldots, \qquad \xi = (n+1/2)\pi \equiv \xi_{n+1/2} \tag{6.22}$$

and the equation for the singular hypersurfaces is of the form

$$\frac{r_g^{1/2}t}{R^{3/2}} = \xi_{n+1/2},\ n = 0, 1, 2, \ldots \tag{6.23}$$

For the singularities (6.18) we have $\sin\eta < 0$ so that

$$\eta = (n+1)2\pi - \beta, \quad 0 < \beta < \pi, \quad n = 0, 1, 2, \ldots \tag{6.24}$$

The equation for β is of the form

$$[(n+1)3\pi - \frac{3}{2}(\beta + \sin\beta)]\sin\beta - (1+\cos\beta)^2 = 0, \quad 0 < \beta < \pi, \quad n = 0, 1, 2, \ldots, \quad \beta = \beta_{n+1} \tag{6.25}$$

Corresponding values of ξ are

$$\xi_{\beta,n+1} = (n+1)\pi - \frac{1}{2}(\beta_{n+1} + \sin\beta_{n+1}), \quad n = 0, 1, 2, \ldots \tag{6.26}$$

and the equation for the singular hypersurfaces is of the form

$$\frac{r_g^{1/2}t}{R^{3/2}} = \xi_{\beta,n+1}, \quad n = 0, 1, 2, \ldots \tag{6.27}$$

Matching metrics
The only parameter that appears in the metric is the Schwarzschild radius r_g. Therefore matching metrics at singularities amounts to taking the same value of the star mass for all the regions of the spacetime.

Black and white regions
According to (6.11) χ changes between values 1 and 0,

$$\chi(\xi_n) = 1, \quad \chi(\xi_{n+1/2}) = 0, \quad \xi_n = n\pi, \quad \xi_{n+1/2} = (n+1/2)\pi, \quad n = 0, 1, 2, \ldots \tag{6.28}$$

and $\chi(\xi)$ decreases for $\xi_n < \xi < \xi_{n+1/2}$ and increases for $\xi_{n+1/2} < \xi < \xi_{n+1}$. Thus, in view of (6.12), the regions of black and white hole are, respectively,

$$\frac{r_g^{1/2}t}{R^{3/2}} = \xi, \quad \text{black}: \ \xi_n < \xi < \xi_{n+1/2}, \quad \text{white}: \ \xi_{n+1/2} < \xi < \xi_{n+1}, \quad n = 0, 1, 2, \ldots \tag{6.29}$$

The singularities (6.16) correspond to passages black→white (black-white singularities), the singularities (6.18) lie in white regions (white singularities).

Geodesic equations
We shall consider radial geodesics:

$$(K^i) = (K^R, 0, 0), \qquad \omega^2 - m^2 = h_{RR}(K^R)^2 \tag{6.30}$$

where by (6.10)

$$h_{RR} = \frac{(\partial_R r)^2}{1 - r_g/R} \tag{6.31}$$

We have

$$\left(\frac{K^R}{K^0}\right)^2 = \frac{\omega^2 - m^2}{\omega^2}\frac{1}{h_{RR}} \tag{6.32}$$

Equations (3.4) boil down to [5]

$$\frac{dK^R}{du} + \Gamma^R_{RR}(K^R)^2 + 2\Gamma^R_{0R}K^0K^R = 0 \tag{6.33}$$

$$\frac{dK^0}{du} = \Gamma^0_{RR}(K^R)^2 = 0 \tag{6.34}$$

with

$$\Gamma^R_{RR} = \frac{1}{2}\frac{\partial_R h_{RR}}{h_{RR}}, \quad \Gamma^R_{0R} = \frac{1}{2}\frac{\partial_t h_{RR}}{h_{RR}}, \quad \Gamma^0_{RR} = \frac{1}{2}\partial_t h_{RR} \tag{6.35}$$

A trivial, familiar solution is

$$K^R = 0, \quad R = \text{const}, \qquad K^0 = \omega = m \neq 0 \tag{6.36}$$

—there is no problem of matching for it.

Matching geodesics at a black-white singularity

In the vicinity of a black-white hole singularity (6.16), (6.17) we make use of equation (6.34). With (6.35), (6.32), and (6.31) we obtain

$$\frac{dK^0}{du} = \frac{\partial_t\partial_R r}{\partial_R r}\frac{(K^0)^2 - m^2}{K^0}\frac{dt}{du} = 0 \tag{6.37}$$

whence

$$\frac{d}{dt}\left[(K^0)^2 - m^2\right] + 2\frac{\partial_t\partial_R r}{\partial_R r}\left[(K^0)^2 - m^2\right] = 0 \tag{6.38}$$

From (6.32) follows

$$\left(\frac{dR}{dt}\right)^2 = \frac{(K^0)^2 - m^2}{(K^0)^2}\frac{1 - r_g/R}{(\partial_R r)^2} \tag{6.39}$$

We find from (6.12), (6.11) in the vicinity of $r = 0$

$$\partial_R r \approx -\left(\frac{3}{2}\right)^{2/3}\xi_{n+1/2}\frac{1}{(\xi - \xi_{n+1/2})^{1/3}}, \qquad \frac{\partial_t\partial_R r}{\partial_R r} \approx -\frac{1}{3}\frac{\xi_{n+1/2}}{t_s}\frac{1}{\xi - \xi_{n+1/2}} \tag{6.40}$$

where $t_s = t_{\text{singularity}}$. By (6.39)

$$\frac{dR}{dt} \approx 0 \tag{6.41}$$

so that

$$\xi - \xi_{n+1/2} \approx \frac{\xi_{n+1/2}}{t_s}(t - t_s) \tag{6.42}$$

Thus we obtain from (6.38)

$$\frac{d}{dt}\left[(K^0)^2 - m^2\right] - \frac{2}{3}\frac{1}{t - t_s}\left[(K^0)^2 - m^2\right] \approx 0 \tag{6.43}$$

from where

$$(K^0)^2 \approx m^2 + A^2(t-t_s)^{2/3} \tag{6.44}$$

Now (6.39) gives

$$\left(\frac{dR}{dt}\right)^2 \approx b^2 \frac{A^2(t-t_s)^{4/3}}{m^2 + A^2(t-t_s)^{2/3}}, \quad b^2 = \left(\frac{2}{3}\right)^{4/3} \frac{R_s^2(1-r_g/R_s)}{r_g^{2/3} t_s^2} \tag{6.45}$$

whence

$$\frac{dR}{dt} \approx |b| \frac{A(t-t_s)^{2/3}}{\sqrt{m^2 + A^2(t-t_s)^{2/3}}} = |b| \left\{ \begin{array}{l} \dfrac{A}{m}(t-t_s)^{2/3}, \ m \neq 0 \\ \\ \dfrac{A}{|A|}|t-t_s|^{1/3}, \ m = 0 \end{array} \right. \equiv \tilde{A}|t-t_s|^{\beta}, \quad 0 < \beta < 1 \tag{6.46}$$

We obtain

$$R - R_s \approx \tilde{A}\frac{1}{1+\beta}|t-t_s|^{1+\beta}\mathrm{sgn}(t-t_s), \quad \frac{d^2R}{dt^2} \approx \tilde{A}\beta\frac{1}{|t-t_s|^{1-\beta}}\mathrm{sgn}(t-t_s) \tag{6.47}$$

In fact, (6.46), (6.47) describe two solutions: for $t < t_s$ and $t > t_s$, so that

$$\tilde{A} = \tilde{A}^{[\mathrm{sgn}(t-t_s)]} \tag{6.48}$$

Thus

$$\frac{d^2R}{dt^2} \approx \tilde{A}^{[\mathrm{sgn}(t-t_s)]}\beta\frac{1}{|t-t_s|^{1-\beta}}\mathrm{sgn}(t-t_s) \tag{6.49}$$

The quantities R and dR/dt are continuous. In order that $R(t)$ be maximally smooth

$$\lim_{\delta\to 0}\frac{(d^2R/dt^2)_{t_s+\delta}}{(d^2R/dt^2)_{t_s-\delta}} = 1 \tag{6.50}$$

should hold, whence

$$-\tilde{A}^{[-1]} = \tilde{A}^{[+1]} \equiv \tilde{A}, \quad \tilde{A}^{[\mathrm{sgn}(t-t_s)]} = \tilde{A}\mathrm{sgn}(t-t_s) \tag{6.51}$$

so that

$$R - R_s \approx \tilde{A}\frac{1}{1+\beta}|t-t_s|^{1+\beta}, \quad \frac{dR}{dt} \approx \tilde{A}|t-t_s|^{\beta}\mathrm{sgn}(t-t_s), \quad \frac{d^2R}{dt^2} \approx \tilde{A}\beta\frac{1}{|t-t_s|^{1-\beta}} \tag{6.52}$$

We find

$$\frac{d^2R}{dt^2} \approx \frac{\beta}{(1+\beta)^{(1-\beta)/(1+\beta)}}\tilde{A}^{2/(1+\beta)}\frac{R-R_s}{(R-R_s)^{2/(1+\beta)}} \tag{6.53}$$

This is the case of repulsion and reflection from the point R_s. The curve $R(t)$ is $\bar{C}^2$.

Matching geodesics at a white singularity: a general solution
In the vicinity of a white singularity (6.18) we make use of equation (6.33):

$$\frac{dK^R}{du} + \left\{\left[\frac{\partial_R\partial_R r}{\partial_R r} + \frac{1}{2}\frac{\partial_R(r_g/R)}{1-r_g/R}\right]K^R + \left[2\frac{\partial_t\partial_R r}{\partial_R r}\right]K^0\right\}\frac{dR}{du} = 0 \tag{6.54}$$

whence, in view of $\partial_R r \approx 0$,

$$\frac{dK^R}{dR} + \frac{\partial_R \partial_R r}{\partial_R r} K^0 + 2\frac{\partial_t \partial_R r}{\partial_R r} K^0 \approx 0 \tag{6.55}$$

Consider a solution for which

$$\frac{K^0}{K^R} \approx 0 \tag{6.56}$$

so that

$$\frac{dt}{dR} \approx 0, \qquad \partial_R r \approx (\partial_R \partial_R r)(R - R_s) \tag{6.57}$$

We obtain

$$\frac{dK^R}{dR} + \frac{K^R}{R - R_s} \approx 0, \quad \frac{d}{dR}\left[(R - R_s)K^R\right] \approx 0 \tag{6.58}$$

from which

$$K^R \approx \frac{B}{R - R_s} \tag{6.59}$$

Next we make use of

$$\frac{dR}{dt} = \frac{K^R}{K^0} \tag{6.60}$$

We find

$$\begin{aligned} \omega = K^0 = \sqrt{m^2 + \frac{(\partial_R r)^2}{1 - r_g/R_s}(K^R)^2} &\approx \sqrt{m^2 + \frac{(\partial_R \partial_R r)^2}{1 - r_g/R_s}\left[(R - R_s)K^R\right]^2} \\ &\approx \sqrt{m^2 + \frac{(\partial_R \partial_R r)^2 B^2}{1 - r_g/R_s}} = \text{const} \end{aligned} \tag{6.61}$$

Thus

$$(R - R_s)dR \approx \frac{B}{\omega}dt, \quad \frac{1}{2}(R - R_s)^2 \approx \frac{B}{\omega}(t - t_s) \tag{6.62}$$

whence it follows that

$$B(t - t_s) \geq 0, \quad B(t - t_s) = |B(t - t_s)| = |B||(t - t_s)|, \quad (R - R_s)^2 \approx \frac{2|B|}{\omega}|t - t_s| \tag{6.63}$$

so that

$$R - R_s \approx \pm\sqrt{\frac{2|B|}{\omega}}|t - t_s|^{1/2} = \pm(\text{sgn}B)[\text{sgn}(t - t_s)]\sqrt{\frac{2|B|}{\omega}}|t - t_s|^{1/2} \tag{6.64}$$

or

$$R - R_s \approx (\text{sgn}B)\sqrt{\frac{2|B|}{\omega}}|t - t_s|^{1/2}\text{sgn}(t - t_s) \tag{6.65}$$

with $\text{sgn}B = \pm 1$ independently of $\text{sgn}(t - t_s)$. Thus we obtain

$$R - R_s \approx B\sqrt{\frac{2}{|B|\omega}}|t - t_s|^{1/2}\text{sgn}(t - t_s) \tag{6.66}$$

Now we have for two solutions

$$R - R_s \approx B^{[\mathrm{sgn}(t-t_s)]}\sqrt{\frac{2}{\omega|B^{[\mathrm{sgn}(t-t_s)]}|}}|t-t_s|^{1/2}\mathrm{sgn}(t-t_s) \tag{6.67}$$

$$\frac{dR}{dt} \approx \frac{1}{\sqrt{2\omega}}\frac{B^{[\mathrm{sgn}(t-t_s)]}}{|B^{[\mathrm{sgn}(t-t_s)]}|^{1/2}}\frac{1}{|t-t_s|^{1/2}} \tag{6.68}$$

From

$$\lim_{\delta\to 0}\frac{(dR/dt)_{t_s+\delta}}{(dR/dt)_{t_s-\delta}} = 1 \tag{6.69}$$

follows

$$B^{[+1]} = B^{[-1]} \equiv B \tag{6.70}$$

Thus we have finally

$$R - R_s \approx \sqrt{\frac{2}{\omega}}\frac{B}{\sqrt{|B|}}|t-t_s|^{1/2}\mathrm{sgn}(t-t_s)$$

$$\frac{dR}{dt} \approx \frac{1}{\sqrt{2\omega}}\frac{B}{\sqrt{|B|}}\frac{1}{|t-t_s|^{1/2}} \tag{6.71}$$

$$\frac{d^2R}{dt^2} \approx -\frac{1}{\sqrt{8\omega}}\frac{B}{\sqrt{|B|}}\frac{\mathrm{sgn}(t-t_s)}{|t-t_s|^{3/2}}$$

We find

$$\frac{d^2R}{dt^2} \approx -\frac{B^2}{\omega^2}\frac{R-R_s}{(R-R_s)^4} \tag{6.72}$$

This is the case of attraction by and passage through the point R_s. The curve $R(t)$ is $\bar{C}^1$.

Matching geodesics at a white singularity: an exceptional solution
Now let us consider the case where

$$\partial_R r \approx 0, \qquad \frac{dt}{dR} \neq 0 \tag{6.73}$$

We shall be based on equations (6.34), (6.32):

$$\frac{dK^0}{du} + \frac{(\partial_R r)(\partial_t \partial_R r)}{1 - r_g/R}K^R\frac{dR}{du} = 0 \tag{6.74}$$

$$\left(\frac{dt}{dR}\right)^2 = \frac{\omega^2}{\omega^2 - m^2}\frac{(\partial_R r)^2}{1 - r_g/R} \tag{6.75}$$

From (6.75), (6.73) follows

$$\omega \approx m \neq 0 \tag{6.76}$$

We find from (6.32)

$$K^R \approx \pm\sqrt{2m(\omega - m)}\frac{\sqrt{1 - r_g/R}}{|\partial_R r|} \tag{6.77}$$

and from (6.74)

$$\frac{d\omega}{dR}+(\mathrm{sgn}K^R)(\mathrm{sgn}\partial_R r)\frac{\partial_t\partial_R r}{\sqrt{1-r_g/R}}\sqrt{2m(\omega-m)}\approx 0 \tag{6.78}$$

whence

$$\omega-m\approx\frac{1}{2}\frac{(\partial_t\partial_R r)^2}{1-r_g/R}(R-R_s)^2,\quad \omega_s=m \tag{6.79}$$

Now (6.75) results in

$$\left(\frac{dt}{dR}\right)^2\approx\frac{(\partial_R r)^2}{(\partial_t\partial_R r)^2(R-R_s)^2} \tag{6.80}$$

We have

$$\partial_R r\approx(\partial_t\partial_R r)(t-t_s)+(\partial_R\partial_R r)(R-R_s)\approx\left[\frac{dt}{dR}+\frac{\partial_R\partial_R r}{\partial_t\partial_R r}\right](\partial_t\partial_R r)(R-R_s) \tag{6.81}$$

From (6.80), (6.81) follows

$$\frac{dt}{dR}\approx-\frac{1}{2}\frac{\partial_R\partial_R r}{\partial_t\partial_R r} \tag{6.82}$$

or, in view of (6.12),

$$\left(\frac{dt}{dR}\right)_s=\frac{3}{4}\frac{t_s}{R_s} \tag{6.83}$$

This gives an exceptional solution, for which there is no problem of matching. We have from (6.81)

$$\left(\frac{dt}{dR}\right)_{\partial_R r=0}=\frac{3}{2}\frac{t_s}{R_s} \tag{6.84}$$

so that

$$\left(\frac{dt}{dR}\right)_s=\frac{1}{2}\left(\frac{dt}{dR}\right)_{\partial_R r=0} \tag{6.85}$$

Asymptotic flatness
We have for $\xi\approx\xi_n$

$$\chi\approx 1,\quad \frac{d\chi}{d\xi}\approx 0,\quad r\approx R,\quad \partial_R r\approx 1 \tag{6.86}$$

which together with $R\gg r_g$ gives for the metric

$$ds^2\approx dt^2-[dR^2+R^2(d\theta^2+\sin^2\theta d\phi^2)],\qquad \frac{r_g^{1/2}t}{R^{3/2}}\approx\pi n,\qquad R\gg r_g \tag{6.87}$$

This holds, specifically, for

$$R\gg(r_g^{1/2}t)^{2/3},\ r_g \tag{6.88}$$

Light cone
We find for the light cone

$$\left|\left(\frac{dt}{dR}\right)_{\mathrm{light}}\right|=\frac{|d\chi/d\xi|}{\sqrt{1-r_g/R}}\left|\frac{3}{2}\xi-\frac{\chi}{d\chi/d\xi}\right| \tag{6.89}$$

and for $r = \text{const}$

$$\left(\frac{dt}{dR}\right)_r = \left(\frac{R}{r_g}\right)^{1/2}\left[\frac{3}{2}\xi - \frac{\chi}{d\chi/d\xi}\right] \tag{6.90}$$

so that

$$\left|\frac{(dt/dR)_{\text{light}}}{(dt/dR)_r}\right| = \left(\frac{1-\chi}{r/r_g - \chi}\right)^{1/2} \tag{6.91}$$

this quantity

$$= \infty \text{ for } r = 0, \quad > 1 \text{ for } 0 < r < r_g, \quad = 1 \text{ for } r = r_g, \quad < 1 \text{ for } r > r_g, \quad = 0 \text{ for } r = R \tag{6.92}$$

A stationary star
For completeness let us touch on the case of a stationary star. Its boundary is determined by

$$r(t, R) = a = \text{const}, \quad R_a = R_a(t) \tag{6.93}$$

the exterior region being

$$r(t, R) > a \tag{6.94}$$

We find with the help of (6.9)

$$\begin{aligned} &\eta_a = \eta_a(t), \quad \eta_a(0) = 0, \quad R_a(0) = a, \quad -\pi < \eta_a(t) < \pi, \quad \lim_{t\to\mp\infty} \eta_a(t) = \mp\pi \\ &a \le R_a(t) < \infty, \quad \lim_{t\to\mp\infty} R_a(t) = +\infty \end{aligned} \tag{6.95}$$

7 A big crunch-bang

Spacetime manifold
Spacetime manifold of the standard model of the universe is the Robertson-Walker spacetime, i.e., a product manifold (4.1), the three-dimensional space S being a sphere, flat Euclidean space, or a hyperboloid.

Synchronous coordinates
Coordinates are t and "spherical" coordinates r, θ, ϕ, where

$$0 \le r \le 1 \text{ for sphere}, \quad 0 \le r < \infty \text{ for flat space and hyperboloid} \tag{7.1}$$

Metric
Metric is of the form

$$\begin{aligned} &ds^2 = dt^2 - a^2(t)\left[\tfrac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \\ &k = 1 \text{ for sphere}, \quad k = 0 \text{ for flat space}, \quad k = 1 \text{ for hyperboloid} \end{aligned} \tag{7.2}$$

Metric singularities

It suffices to consider one singular hypersurface:

$$t = 0, \qquad a(0) = 0 \tag{7.3}$$

in the vicinity of which

$$a(t) \approx b|t|^{\beta}, \quad 0 < \beta < 1 \tag{7.4}$$

$t < 0$ and $t > 0$ corresponding to the contracting and expanding universe respectively. This is a crunch-bang singularity.

Matching metrics
Matching metrics amounts to taking the same values of b, β for $t > 0$ and $t < 0$.

Geodesic equations
We use relations (3.4) through (3.6) for geodesics and consider "radial" geodesics:

$$(K^i) = (K^r, 0, 0), \qquad (K^0)^2 = h_{rr}(K^r)^2 + m^2 \tag{7.5}$$

where by (7.2)

$$h_{rr} = \frac{a^2}{1 - kr^2} \tag{7.6}$$

Equations (3.4) boil down to [5]

$$\frac{dK^r}{du} + \Gamma^r_{rr}(K^r)^2 + 2\Gamma^r_{0r}K^0K^r = 0 \tag{7.7}$$

$$\frac{dK^0}{du} + \Gamma^0_{rr}(K^r)^2 = 0 \tag{7.8}$$

with

$$\Gamma^r_{rr} = \frac{kr}{1 - kr^2}, \quad \Gamma^r_{0r} = \frac{\dot{a}}{a}, \quad \Gamma^0_{rr} = \frac{a\dot{a}}{1 - kr^2} \tag{7.9}$$

A trivial solution is

$$K^r = 0, \quad r = \text{const}, \qquad \omega = K^0 = m \neq 0 \tag{7.10}$$

with no problem of matching.

Matching geodesics
In the vicinity of the crunch-bang singularity (7.3), (7.4) we make use of equation (7.7):

$$\frac{dK^r}{du} + \left[\frac{kr}{1 - kr^2}K^r + 2\frac{\beta}{|t|}(\mathrm{sgn}t)K^0\right]K^r = 0 \tag{7.11}$$

From (7.5), (7.6) follows

$$K^0 \geq \frac{b|t|^{\beta}}{\sqrt{1 - kr^2}}|K^r| \tag{7.12}$$

so that we obtain from (7.11)

$$\frac{dK^r}{du} + \frac{2\beta}{t}K^rK^0 \approx 0 \tag{7.13}$$

or

$$\frac{dK^r}{K^r} + 2\beta \frac{dt}{t} \approx 0 \tag{7.14}$$

whence

$$|K^r| \approx \frac{|A|}{|t|^{2\beta}}, \qquad K^r \approx \frac{A}{|t|^{2\beta}} \tag{7.15}$$

Now from (7.5) follows

$$(K^0)^2 \approx \frac{b^2 A^2}{(1-kr^2)|t|^{2\beta}} + m^2 \approx \frac{b^2 A^2}{(1-kr^2)|t|^{2\beta}}, \qquad K^0 \approx \frac{b|A|}{\sqrt{1-kr^2}|t|^{\beta}} \tag{7.16}$$

We have

$$\frac{dr}{dt} = \frac{K^r}{K^0} \approx (\mathrm{sgn}A)\frac{\sqrt{1-kr^2}}{b}\frac{1}{|t|^{\beta}} \tag{7.17}$$

from where

$$r - r_0 \approx (\mathrm{sgn}A)\frac{\sqrt{1-kr_0^2}}{b}\frac{1}{1-\beta}(\mathrm{sgn}t)|t|^{1-\beta}, \quad r_0 = r(t=0) \tag{7.18}$$

Now we have for two solutions

$$(r-r_0)^{[\mathrm{sgn}t]} \approx (\mathrm{sgn}A^{[\mathrm{sgn}t]})\frac{\sqrt{1-kr_0^2}}{b(1-\beta)}(\mathrm{sgn}t)|t|^{1-\beta} \tag{7.19}$$

$$\left(\frac{dr}{dt}\right)^{[\mathrm{sgn}t]} \approx (\mathrm{sgn}A^{[\mathrm{sgn}t]})\frac{\sqrt{1-kr_0^2}}{b}\frac{1}{|t|^{\beta}} \tag{7.20}$$

From the condition

$$\lim_{\delta\to 0}\frac{(dr/dt)_{\delta}}{(dr/dt)_{-\delta}} = 1 \tag{7.21}$$

follows

$$\mathrm{sgn}A^{[+1]} = \mathrm{sgn}A^{[-1]} \equiv \mathrm{sgn}A \tag{7.22}$$

and we may put

$$A^{[+1]} = A^{[-1]} \equiv A \tag{7.23}$$

Thus we have finally

$$r - r_0 \approx (\mathrm{sgn}A)\frac{\sqrt{1-kr_0^2}}{b(1-\beta)}(\mathrm{sgn}t)|t|^{1-\beta}$$

$$\frac{dr}{dt} \approx (\mathrm{sgn}A)\frac{\sqrt{1-kr_0^2}}{b}\frac{1}{|t|^{\beta}} \tag{7.24}$$

$$\frac{d^2r}{dt^2} \approx -(\mathrm{sgn}A)\frac{\beta\sqrt{1-kr_0^2}}{b}\frac{\mathrm{sgn}t}{|t|^{1+\beta}}$$

We find

$$\frac{d^2r}{dt^2} \approx -\beta(1-\beta)\left(\frac{\sqrt{1-kr_0^2}}{1-\beta}\right)^{2/(1-\beta)}\frac{r-r_0}{|r-r_0|^{2/(1-\beta)}} \tag{7.25}$$

This is the case of attraction by and passage through the point r_0.

The curve $r(t)$ is $\bar{C}^1$.

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